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Some subclasses of multivalent spirallike meromorphic functions

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Dedicated to Professor Hari M Srivastava

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Abstract
In the present paper, we introduce and investigate two new subclasses $\mathcal{MS}_p(\alpha,\beta)$ and $\mathcal{MC}_p(\alpha,\beta)$ of meromorphic functions. Such results as integral representations and coefficient inequalities are proved. The results presented here would provide extensions of those given in earlier works.
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1 Introduction

Let Σ_p denote the class of functions f of the form

$$f(z) = z^{-p} + \sum_{n=1-p}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

Let $\mathcal{P}$ denote the class of functions p given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathbb{U}),$$

which are analytic in $\mathbb{U}$ and satisfy the condition

$$\Re\bigl(p(z)\bigr) > 0 \quad (z \in \mathbb{U}).$$

A function $f \in \Sigma_p$ is said to be in the class $\mathcal{MS}_p(\alpha)$ of meromorphic p-valent starlike functions of order α if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in \mathbb{U}; 0 \leqq \alpha < p). \tag{1.2}$$

Moreover, a function $f \in \Sigma_p$ is said to be in the class $\mathcal{MK}_p(\alpha)$ of meromorphic p-valent convex functions of order α if it satisfies the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (z \in \mathbb{U}; 0 \leqq \alpha < p). \tag{1.3}$$

It is readily verified from (1.2) and (1.3) that

$$f \in \mathcal{MK}_p(\alpha) \quad \Longleftrightarrow \quad -\frac{zf'}{p} \in \mathcal{MS}_p^*(\alpha).$$

In [1], Wang *et al.* introduced and investigated two new subclasses of the class Σ_p. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{M}_p(\beta)$ if it is characterized by the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > -\beta \quad (z \in \mathbb{U}; \beta > p).$$

Also, a function $f \in \Sigma_p$ is said to be in the class $\mathcal{N}_p(\beta)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\beta \quad (z \in \mathbb{U}; \beta > p).$$

Let $\mathcal{A}_p$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

which are analytic in $\mathbb{U}$. If it satisfies the condition

$$\Re\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) < \beta \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}; \beta > p\cos\alpha\right),$$

then we say that $f \in \mathcal{S}_p(\alpha, \beta)$. Furthermore, let $\mathcal{C}_p(\alpha, \beta)$ denote the subclass of $\mathcal{A}_p$ consisting of functions which satisfy the inequality

$$\Re\left(e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) < \beta \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}; \beta > p\cos\alpha\right).$$

The function classes $\mathcal{S}_p(\alpha, \beta)$ and $\mathcal{C}_p(\alpha, \beta)$ were introduced and studied recently by Uyanik *et al.* [2].

Motivated essentially by the above mentioned work, we introduce and investigate the following two subclasses of the class Σ_p of meromorphic functions.

Definition 1 A function $f \in \Sigma_p$ is said to be in the class $\mathcal{MS}_p(\alpha, \beta)$ if it satisfies the condition

$$\Re\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > -\beta \quad (z \in \mathbb{U}) \tag{1.4}$$

for some real α and β, where (and throughout this paper unless otherwise mentioned) the parameters α and β are constrained as follows:

$$|\alpha| < \frac{\pi}{2} \quad \text{and} \quad \beta > p\cos\alpha.$$

Furthermore, a function $f \in \Sigma_p$ is said to be in the class $\mathcal{MC}_p(\alpha,\beta)$ if it satisfies the inequality

$$\Re\left(e^{i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right) > -\beta \quad (z \in \mathbb{U}). \tag{1.5}$$

Remark 1 Taking $\alpha = 0$, we get the function classes introduced by Wang *et al.* [1].

Remark 2 We note that $f \in \mathcal{MS}_p(\alpha,\beta)$ if and only if

$$-e^{i\alpha}\frac{zf'(z)}{f(z)} \prec \frac{pe^{i\alpha} - (2\beta - pe^{-i\alpha})z}{1-z}. \tag{1.6}$$

Also, $f \in \mathcal{MC}_p(\alpha,\beta)$ if and only if

$$-e^{i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{pe^{i\alpha} - 2(\beta - pe^{-i\alpha})z}{1-z}. \tag{1.7}$$

For some investigations of meromorphic functions, see (for example) the works [1, 3–10] and the references cited in.

In the present paper, we aim at proving some interesting properties such as integral representations and coefficient inequalities of the function classes $\mathcal{MS}_p(\alpha,\beta)$ and $\mathcal{MC}_p(\alpha,\beta)$.

2 Main results

We begin by presenting an integral representation of functions belonging to the class $\mathcal{MS}_p(\alpha,\beta)$.

Theorem 1 *Let $f \in \mathcal{MS}_p(\alpha,\beta)$. Then*

$$f(z) = z^{-p}\cdot\exp\left(2(\beta - p\cos\alpha)e^{-i\alpha}\int_0^z \frac{\omega(t)}{t(1-\omega(t))}\,dt\right) \quad \left(z \in \mathbb{U}^*\right), \tag{2.1}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Proof For $f \in \mathcal{MS}_p(\alpha,\beta)$, we know that (1.6) holds true. It follows that

$$-e^{i\alpha}\frac{zf'(z)}{f(z)} = pe^{i\alpha} - \frac{2(\beta - p\cos\alpha)\omega(z)}{1-\omega(z)}, \tag{2.2}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$. We next find from (2.2) that

$$\frac{f'(z)}{f(z)} + \frac{p}{z} = \frac{2(\beta - p\cos\alpha)e^{-i\alpha}\omega(z)}{z(1-\omega(z))} \quad \left(z \in \mathbb{U}^*\right), \tag{2.3}$$

which, upon integration, yields

$$\log\bigl(z^p f(z)\bigr) = 2(\beta - p\cos\alpha)e^{-i\alpha}\int_0^z \frac{\omega(t)}{t(1-\omega(t))}\,dt. \tag{2.4}$$

The assertion (2.1) of Theorem 1 can be easily derived from (2.4). □

Note that $f \in \mathcal{MS}_p(\alpha,\beta)$ if and only if

$$-\frac{zf'(z)}{p} \in \mathcal{MC}_p(\alpha,\beta),$$

we get the following result.

Corollary 1 *Let $f \in \mathcal{MC}_p(\alpha,\beta)$. Then*

$$f(z) = -p\int_{z_0}^z u^{-p-1}\cdot \exp\left(2(\beta - p\cos\alpha)e^{-i\alpha}\int_0^u \frac{\omega(t)}{t(1-\omega(t))}\,dt\right)du \quad \bigl(z\in\mathbb{U}^*\bigr),$$

where ω is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$.

Next, we discuss the coefficient estimates of functions belonging to the classes $\mathcal{MS}_p(\alpha,\beta)$ and $\mathcal{MC}_p(\alpha,\beta)$. The following lemma will be required in the proof of Theorem 2.

Lemma 1 *Let $p\in\mathbb{N}$. Suppose also that the sequence $\{A_{p+m}\}_{m=0}^\infty$ is defined by*

$$\begin{cases} A_p = \frac{\beta - p\cos\alpha}{p} & (m=0), \\ A_{p+m} = \frac{2(\beta-p\cos\alpha)}{2p+m}\bigl(1+\sum_{k=0}^{m-1}A_{p+k}\bigr) & (m\in\mathbb{N}). \end{cases} \tag{2.5}$$

Then

$$A_{p+m} = \frac{2(\beta - p\cos\alpha)}{2\beta + m + 2p - 2p\cos\alpha}\prod_{k=0}^{m}\frac{2\beta + k + 2p - 2p\cos\alpha}{2p+k}$$
$$\bigl(m\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}\bigr). \tag{2.6}$$

Proof By virtue of (2.5), we get

$$(2p+m+1)A_{p+m+1} = 2(\beta - p\cos\alpha)\left(1+\sum_{k=0}^{m}A_{p+k}\right), \tag{2.7}$$

and

$$(2p+m)A_{p+m} = 2(\beta - p\cos\alpha)\left(1+\sum_{k=0}^{m-1}A_{p+k}\right). \tag{2.8}$$

Combining (2.7) and (2.8), we find that

$$\frac{A_{p+m+1}}{A_{p+m}} = \frac{2\beta + m + 2p - 2p\cos\alpha}{2p+m+1} \quad (m\in\mathbb{N}_0). \tag{2.9}$$

Thus,

$$\begin{aligned}
A_{p+m} &= \frac{A_{p+m}}{A_{p+m-1}} \cdot \frac{A_{p+m-1}}{A_{p+m-2}} \cdots \frac{A_{p+1}}{A_p} \cdot A_p \\
&= \frac{2\beta + m - 1 + 2p - 2p\cos\alpha}{2p+m} \cdots \frac{2\beta + 2p - 2p\cos\alpha}{2p+1} \cdot \frac{2\beta - 2p\cos\alpha}{2p} \\
&= \frac{2(\beta - p\cos\alpha)}{2\beta + m + 2p - 2p\cos\alpha} \prod_{k=0}^{m} \frac{2\beta + k + 2p - 2p\cos\alpha}{2p+k} \quad (m \in \mathbb{N}). \qquad (2.10)
\end{aligned}$$

The proof of Lemma 1 is thus completed. □

Theorem 2 *Let* $f(z) = z^{-p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MS}_p(\alpha, \beta)$. *Then*

$$|a_{p+m}| \leqq \frac{2(\beta - p\cos\alpha)}{2\beta + m + 2p - 2p\cos\alpha} \prod_{k=0}^{m} \frac{2\beta + k + 2p - 2p\cos\alpha}{2p+k} \quad (m \in \mathbb{N}_0). \qquad (2.11)$$

Proof Let

$$h(z) := \frac{\beta + e^{i\alpha}\frac{zf'(z)}{f(z)} + ip\sin\alpha}{\beta - p\cos\alpha} \quad \bigl(z \in \mathbb{U}; f \in \mathcal{MS}_p(\alpha, \beta)\bigr). \qquad (2.12)$$

We know that $h \in \mathcal{P}$. It follows that

$$e^{i\alpha} zf'(z) = (\beta - p\cos\alpha) f(z) h(z) - (\beta + ip\sin\alpha) f(z). \qquad (2.13)$$

Suppose that

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots. \qquad (2.14)$$

Then

$$\begin{aligned}
&e^{i\alpha}\bigl(-pz^{-p} + pa_p z^p + (p+1)a_{p+1}z^{p+1} + \cdots + (p+m)a_{p+m}z^{p+m} + \cdots\bigr) \\
&\quad = (\beta - p\cos\alpha)\bigl(z^{-p} + a_p z^p + a_{p+1}z^{p+1} + \cdots\bigr) \times \bigl(1 + h_1 z + h_2 z^2 + \cdots\bigr) \\
&\qquad - (\beta + ip\sin\alpha)\bigl(z^{-p} + a_p z^p + a_{p+1}z^{p+1} + \cdots + a_{p+m}z^{p+m} + \cdots\bigr). \qquad (2.15)
\end{aligned}$$

By evaluating the coefficient of z^{p+m} on both sides of (2.15), we get

$$\begin{aligned}
e^{i\alpha}(p+m)a_{p+m} &= (\beta - p\cos\alpha)(h_{2p+m} + a_p h_m + a_{p+1}h_{m-1} + \cdots + a_{p+m}) \\
&\quad - (\beta + ip\sin\alpha)a_{p+m}. \qquad (2.16)
\end{aligned}$$

On the other hand, it is well known that

$$|h_k| \leqq 2 \quad (k \in \mathbb{N}). \qquad (2.17)$$

From (2.16) and (2.17), we easily get

$$|a_p| \leqq \frac{\beta - p\cos\alpha}{p} \qquad (2.18)$$

and

$$|a_{p+m}| \leqq \frac{2(\beta - p\cos\alpha)}{2p+m}\left(1 + \sum_{k=0}^{m-1}|a_{p+k}|\right). \tag{2.19}$$

Suppose that $p \in \mathbb{N}$. We define the sequence $\{A_{p+m}\}_{m=0}^{\infty}$ as follows:

$$\begin{cases} A_p = \frac{\beta - p\cos\alpha}{p} & (m = 0), \\ A_{p+m} = \frac{2(\beta - p\cos\alpha)}{2p+m}(1 + \sum_{k=0}^{m-1} A_{p+k}) & (m \geqq 1). \end{cases} \tag{2.20}$$

In order to prove that

$$|a_{p+m}| \leqq A_{p+m} \quad (m \in \mathbb{N}_0), \tag{2.21}$$

we use the principle of mathematical induction. It is easy to verify that

$$|a_p| \leqq A_p = \frac{\beta - p\cos\alpha}{p}. \tag{2.22}$$

Thus, assuming that

$$|a_{p+j}| \leqq A_{p+j} \quad (j = 0, 1, \ldots, m; m \in \mathbb{N}_0), \tag{2.23}$$

we find from (2.19) and (2.23) that

$$\begin{aligned} |a_{p+m+1}| &\leqq \frac{2(\beta - p\cos\alpha)}{2p+m+1}\left(1 + \sum_{k=0}^{m}|a_{p+k}|\right) \\ &\leqq \frac{2(\beta - p\cos\alpha)}{2p+m+1}\left(1 + \sum_{k=0}^{m}|A_{p+k}|\right) \\ &= A_{p+m+1} \quad (m \in \mathbb{N}_0). \end{aligned} \tag{2.24}$$

Therefore, by the principle of mathematical induction, we have

$$|a_{p+m}| \leqq A_{p+m} \quad (m \in \mathbb{N}_0). \tag{2.25}$$

By means of Lemma 1 and (2.20), we know that

$$A_{p+m} = \frac{2(\beta - p\cos\alpha)}{2\beta + m + 2p - 2p\cos\alpha}\prod_{k=0}^{m}\frac{2\beta + k + 2p - 2p\cos\alpha}{2p+k} \quad (m \in \mathbb{N}_0). \tag{2.26}$$

Combining (2.25) and (2.26), we readily get the coefficient estimates (2.11) asserted by Theorem 2. □

From Theorem 2, we easily get the following result.

Corollary 2 *Let* $f(z) = z^{-p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MC}_p(\alpha, \beta)$. *Then*

$$|a_{p+m}| \leqq \frac{2p(\beta - p\cos\alpha)}{(p+m)(2\beta + m + 2p - 2p\cos\alpha)} \prod_{k=0}^{m} \frac{2\beta + k + 2p - 2p\cos\alpha}{2p + k} \quad (m \in \mathbb{N}_0).$$

Remark 3 By setting $\alpha = 0$ in Theorem 2, we get the corresponding result due to Wang *et al.* [1].

Theorem 3 *If* $f \in \mathcal{MS}_p(\alpha, \beta)$, *then*

$$\frac{p\cos\alpha - (2\beta - p\cos\alpha)r}{1-r} \leqq \Re\left(-e^{i\alpha} \frac{zf'(z)}{f(z)}\right) \leqq \frac{p\cos\alpha + (2\beta - p\cos\alpha)r}{1+r} \tag{2.27}$$

for $|z| = r < 1$.

Proof Consider the function φ defined by

$$\varphi(z) := \frac{pe^{i\alpha} - (2\beta - pe^{-i\alpha})z}{1-z} \quad (z \in \mathbb{U}). \tag{2.28}$$

Let $z = re^{i\theta}$ $(0 < r < 1)$, we see that

$$\Re\big(\varphi(z)\big) = p\cos\alpha - \frac{2(\beta - p\cos\alpha)r(\cos\theta - r)}{1 + r^2 - 2r\cos\theta}. \tag{2.29}$$

Suppose

$$\psi(t) := p\cos\alpha - \frac{2(\beta - p\cos\alpha)r(t-r)}{1 + r^2 - 2rt} \quad (t := \cos\theta), \tag{2.30}$$

we easily find that

$$\psi'(t) = -2(\beta - p\cos\alpha) \cdot \frac{1 - r^2}{(1 + r^2 - 2rt)^2} > 0. \tag{2.31}$$

This implies

$$p\cos\alpha - \frac{2(\beta - p\cos\alpha)r}{1-r} \leqq \Re\big(\varphi(z)\big) \leqq p\cos\alpha + \frac{2(\beta - p\cos\alpha)r}{1+r}, \tag{2.32}$$

which is equivalent to

$$\frac{p\cos\alpha - (2\beta - p\cos\alpha)r}{1-r} \leqq \Re\big(\varphi(z)\big) \leqq \frac{p\cos\alpha + (2\beta - p\cos\alpha)r}{1+r}. \tag{2.33}$$

Noting that $-e^{i\alpha}\frac{zf'(z)}{f(z)} \prec \varphi(z)$ and $\varphi(z)$ is univalent in $\mathbb{U}$, we prove the inequality (2.27). □

Taking $\alpha = 0$ in Theorem 3, we have the following corollary.

Corollary 3 *If $f \in \mathcal{MS}_p(0,\beta)$, then*

$$\frac{p-(2\beta-p)r}{1-r} \leqq \Re\left(\frac{zf'(z)}{f(z)}\right) \leqq \frac{p+(2\beta-p)r}{1+r}$$

for $|z| = r < 1$.

Similar to the proof of Theorem 3, we get the following result.

Corollary 4 *If $f \in \mathcal{MC}_p(\alpha,\beta)$, then*

$$\frac{p\cos\alpha-(2\beta-p\cos\alpha)r}{1-r} \leqq \Re\left(-e^{i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right) \leqq \frac{p\cos\alpha+(2\beta-p\cos\alpha)r}{1+r}$$

for $|z| = r < 1$.

Corollary 5 *If $f \in \mathcal{MC}_p(0,\beta)$, then*

$$\frac{p-(2\beta-p)r}{1-r} \leqq \Re\left(1+\frac{zf''(z)}{f'(z)}\right) \leqq \frac{p+(2\beta-p)r}{1+r}$$

for $|z| = r < 1$.

Now, we present some sufficient conditions for functions belonging to the classes $\mathcal{MS}_p(\alpha,\beta)$ and $\mathcal{MC}_p(\alpha,\beta)$.

Theorem 4 *If $f \in \mathcal{MS}_p(\alpha,\beta)$ satisfies the condition*

$$\sum_{n=1-p}^{\infty}\left(\left|ne^{i\alpha}+\lambda\right|+\left|ne^{i\alpha}+2\beta-\lambda\right|\right)|a_n| \leqq \left|pe^{i\alpha}-2\beta+\lambda\right|-\left|pe^{i\alpha}-\lambda\right| \tag{2.34}$$

for some real α, β and λ ($0 \leqq \lambda \leqq p\cos\alpha$), then $f \in \mathcal{MS}_p(\alpha,\beta)$.

Proof To prove $f \in \mathcal{MS}_p(\alpha,\beta)$, it suffices to show that

$$\left|\frac{e^{i\alpha}\frac{zf'(z)}{f(z)}+\lambda}{e^{i\alpha}\frac{zf'(z)}{f(z)}+(2\beta-\lambda)}\right| < 1 \quad (z \in \mathbb{U}; 0 \leqq \lambda \leqq p\cos\alpha). \tag{2.35}$$

From (2.34), we know that

$$\left|pe^{i\alpha}-2\beta+\lambda\right|-\sum_{n=1-p}^{\infty}\left|ne^{i\alpha}+2\beta-\lambda\right||a_n| \geqq \left|pe^{i\alpha}-\lambda\right|+\sum_{n=1-p}^{\infty}\left|ne^{i\alpha}+\lambda\right||a_n| > 0. \tag{2.36}$$

Now, by the maximum modulus principle, we deduce from (1.1) and (2.36) that

$$\begin{aligned}\left|\frac{e^{i\alpha}\frac{zf'(z)}{f(z)}+\lambda}{e^{i\alpha}\frac{zf'(z)}{f(z)}+(2\beta-\lambda)}\right| &= \left|\frac{(-pe^{i\alpha}+\lambda)+\sum_{n=1-p}^{\infty}(ne^{i\alpha}+\lambda)a_n z^{n+p}}{(-pe^{i\alpha}+2\beta-\lambda)+\sum_{n=1-p}^{\infty}(ne^{i\alpha}+2\beta-\lambda)a_n z^{n+p}}\right| \\ &< \frac{|pe^{i\alpha}-\lambda|+\sum_{n=1-p}^{\infty}|ne^{i\alpha}+\lambda||a_n|}{|pe^{i\alpha}-2\beta+\lambda|-\sum_{n=1-p}^{\infty}|ne^{i\alpha}+2\beta-\lambda||a_n|} \\ &\leqq 1. \end{aligned} \tag{2.37}$$

Therefore, if f satisfies the coefficient estimate (2.34), then we know that f satisfies the inequality (2.35). This completes the proof of Theorem 4. □

Corollary 6 *If $f \in \mathcal{MC}_p(\alpha,\beta)$ satisfies the inequality*

$$\sum_{n=1-p}^{\infty}|n|\big(\left|ne^{i\alpha}+\lambda\right|+\left|ne^{i\alpha}+2\beta-\lambda\right|\big)|a_n| \leqq p\big(\left|pe^{i\alpha}-2\beta+\lambda\right|-\left|pe^{i\alpha}-\lambda\right|\big)$$

for some real α, β and λ $(0 \leqq \lambda \leqq p\cos\alpha)$, then $f \in \mathcal{MC}_p(\alpha,\beta)$.

We need the following lemma to prove our next theorem.

Lemma 2 (See [11]) *Let φ be a nonconstant regular function in $\mathbb{U}$. If $|\varphi|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0\varphi'(z_0) = k\varphi(z_0),$$

where $k \geqq 1$ is a real number.

Theorem 5 *If $f \in \mathcal{MS}_p(0,\beta)$ satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| < \frac{\beta-p}{2\beta} \quad (z \in \mathbb{U}) \tag{2.38}$$

for some real $\beta > p$, then $f \in \mathcal{MS}_p(0,\beta)$.

Proof Let us define the function ϕ by

$$\phi(z) := \frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}+2\beta-p} \quad (z \in \mathbb{U}), \tag{2.39}$$

then we see that ϕ is analytic in $\mathbb{U}$ and $\phi(0) = 0$. It follows from (2.39) that

$$\frac{zf'(z)}{f(z)} = \frac{-p+(2\beta-p)\phi(z)}{1-\phi(z)}. \tag{2.40}$$

Differentiating both sides of (2.40) logarithmically, we obtain

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} = \frac{(2\beta-p)z\phi'(z)}{-p+(2\beta-p)\phi(z)}+\frac{z\phi'(z)}{1-\phi(z)}. \tag{2.41}$$

By virtue of (2.38) and (2.41), we find that

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| = \left|\frac{2(\beta-p)z\phi'(z)}{[-p+(2\beta-p)\phi(z)][1-\phi(z)]}\right| < \frac{\beta-p}{2\beta}. \tag{2.42}$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|\leqq|z_0|}\left|\phi(z)\right| = \left|\phi(z_0)\right| = 1.$$

Then, Lemma 2 gives us that $\phi(z_0) = e^{i\theta}$ and $z_0\phi'(z_0) = ke^{i\theta}$ $(k \geqq 1)$. For such a point z_0, we have that

$$\begin{aligned}\left|1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right| &= \left|\frac{2(\beta-p)ke^{i\theta}}{[-p+(2\beta-p)e^{i\theta}][1-e^{i\theta}]}\right| \\ &= \frac{2(\beta-p)k}{\sqrt{p^2+(2\beta-p)^2-2p(2\beta-p)\cos\theta}\sqrt{2-2\cos\theta}} \\ &\geqq \frac{\beta-p}{2\beta}.\end{aligned} \tag{2.43}$$

This contradicts our condition (2.38). Therefore, there is no $z_0 \in \mathbb{U}$ such that $|\phi(z_0)| = 1$. This implies that $|\phi(z)| < 1$ $(z \in \mathbb{U}^*)$, that is,

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}+(2\beta-p)}\right| < 1 \quad (z \in \mathbb{U}).$$

Thus, we conclude that $f \in \mathcal{MS}_p(0,\beta)$. $\square$

Theorem 6 *If $f \in \mathcal{MS}_p(0,\beta)$ for some real $p < \beta \leqq p+\frac{1}{2}$, then*

$$\Re\left(\frac{1}{z^pf(z)}\right) > \frac{1}{1-2\beta+2p} \quad (z \in \mathbb{U}). \tag{2.44}$$

Proof Consider the function η such that

$$\frac{1}{z^pf(z)} = \frac{1+(1-2\gamma)\eta(z)}{1-\eta(z)} \tag{2.45}$$

for $\gamma = \frac{1}{1-2\beta+2p}$ and $f(z) \in \mathcal{MS}_p(0,\beta)$. Then we know that

$$\Re\left(-\frac{zf'(z)}{f(z)}\right) = \Re\left(p+\frac{(1-2\gamma)z\eta'(z)}{1+(1-2\gamma)\eta(z)}+\frac{z\eta'(z)}{1-\eta(z)}\right) < \beta. \tag{2.46}$$

Since $\eta(z)$ is analytic in $\mathbb{U}$ and $\eta(0) = 0$, we suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|\leqq|z_0|}\left|\eta(z)\right| = \left|\eta(z_0)\right| = 1.$$

Then, applying Lemma 2, we can write that $\eta(z_0) = e^{i\theta}$ and $z_0\eta'(z_0) = ke^{i\theta}$ $(k \geqq 1)$. This gives us that

$$\begin{aligned}\Re\left(-\frac{z_0 f'(z_0)}{f(z_0)}\right) &= \Re\left(p + \frac{(1-2\gamma)ke^{i\theta}}{1+(1-2\gamma)e^{i\theta}} + \frac{ke^{i\theta}}{1-e^{i\theta}}\right)\\ &\geqq p - \frac{(1-2\gamma)k}{2\gamma} - \frac{k}{2}\\ &\geqq p + \frac{\gamma-1}{2\gamma} = \beta,\end{aligned} \tag{2.47}$$

which contradicts the inequality (2.46). Therefore, there is no $z_0 \in \mathbb{U}$ such that $|\eta(z_0)| = 1$. This means that $|\eta(z)| < 1$, and that

$$\Re\left(\frac{1}{z^p f(z)}\right) > \frac{1}{1-2\beta+2p} \quad (z \in \mathbb{U}). \tag{2.48}$$

The proof of Theorem 6 is thus completed. □

In view of Theorem 6, we get the following result.

Corollary 7 *If $f \in \mathcal{MC}_p(0,\beta)$ for some real $p < \beta \leqq p + \frac{1}{2}$, then*

$$\Re\left(\frac{p}{z^{p+1} f'(z)}\right) > \frac{1}{1-2\beta+2p} \quad (z \in \mathbb{U}).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors jointly worked on deriving the results and approved the final manuscript.

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References

1. Wang, Z-G, Sun, Y, Zhang, Z-H: Certain classes of meromorphic multivalent functions. Comput. Math. Appl. **58**, 1408-1417 (2009)
2. Uyanik, N, Shiraishi, H, Owa, S, Polatoglu, Y: Reciprocal classes of p-valently spirallike and p-valently Robertson functions. J. Inequal. Appl. **2011**, Article ID 61 (2011)
3. Ali, RM, Ravichandran, V: Classes of meromorphic α-convex functions. Taiwan. J. Math. **14**, 1479-1490 (2010)
4. Aouf, MK: Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function. Comput. Math. Appl. **55**, 494-509 (2008)
5. Aouf, MK: Certain subclasses of meromorphically p-valent functions with positive or negative coefficients. Math. Comput. Model. **47**, 997-1008 (2008)
6. Silverman, H, Suchithra, K, Stephen, BA, Gangadharan, A: Coefficient bounds for certain classes of meromorphic functions. J. Inequal. Appl. **2008**, Article ID 931981 (2008)
7. Srivastava, HM, Owa, S: Current Topics in Analytic Function Theory. World Scientific, Singapore (1992)

8. Srivastava, HM, Patel, J: Some subclasses of meromorphically multivalent functions associated with a linear operator. Appl. Math. Comput. **195**, 11-23 (2008)
9. Wang, Z-G, Jiang, Y-P, Srivastava, HM: Some subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function. Comput. Math. Appl. **57**, 571-586 (2009)
10. Wang, Z-G, Liu, Z-H, Xiang, R-G: Some criteria for meromorphic multivalent starlike functions. Appl. Math. Comput. **218**, 1107-1111 (2011)
11. Jack, IS: Functions starlike and convex of order α. J. Lond. Math. Soc. **3**, 469-474 (1971)

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